JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **150**, 159–160 (1990)

A Trace Inequality for Operators

DINESH SINGH

Department of Mathematics, University of Delhi, Delhi 110007, India

Submitted by E. Stanley Lee

Received October 24, 1988

This note provides an infinite-dimensional generalization of Yisong Yang's affirmative answer [3] to the following question of Bellman [1]: Is there a matrix analogue of the arithmetic mean–geometric mean inequality? We show below that this inequality can be extended to the Von Neumann–Schatten class of operators C_1 on an infinite-dimensional separable Hilbert space H. *Our proof provides a shorter and non-computational proof of Yang's theorem, and uses well-known properties of the trace.*

C_p $(1 \leqslant p < \infty)$ stands for the class of all bounded operators A on H such that

$$\sum_{n=1}^{\infty} |\langle Ae_n, e_n \rangle|^p < \infty$$

for each orthonormal basis $\{e_n\}_1^\infty$ in H. Clearly C_1 is contained in C_2. Let $\{e_n\}$ be any orthonormal basis in H. Let Tr: $C_1 \to C$ (complex numbers) be defined by

$$\operatorname{Tr}(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle.$$

It is easy to see that TR is independent of $\{e_n\}$ [2, Lemma 2.2.1], and since C_1 consists of compact operators [2, Theorem 2.1.6], $\operatorname{Tr}(A)$ is the sum of the eigenvalues of A. Further Tr defines an inner product on C_2 given by

$$\langle A, B \rangle = \operatorname{Tr}(B^* A),$$

where B^* is the adjoint of B. This inner product makes C_2 into a Hilbert space; see [2, Theorem 2.4.2].

We now state and prove our result.

0022-247X/90 $3.00

THEOREM. *Let A, B be two positive definite operators in C_1; then*

(1) $\operatorname{Tr}(AB) > 0$ *and*

(2) $\sqrt{\operatorname{Tr}(AB)} < (\operatorname{Tr}(A) + \operatorname{Tr}(B))/2$.

Proof. Choose an orthonormal basis $\{e_n\}_1^\infty$ of H such that each e_n is an eigenvector for A with corresponding eigenvalue α_n. Since $A > 0$ each $\alpha_n > 0$. Let $\beta_n = \{Be_n, e_n\rangle$. Then $\beta_n > 0$ as $B > 0$. Now $\operatorname{Tr}(AB) = \sum_{n=1}^\infty \langle ABe_n, e_n\rangle = \sum_{n=1}^\infty \langle Be_n, Ae_n\rangle = \sum_{n=1}^\infty \alpha_n \langle Be_n, e_n\rangle = \sum_{n=1}^\infty \alpha_n \beta_n > 0$ so that (1) is proved. We now prove (2). Since C_1 is in C_2 we have by the Cauchy–Schwartz inequality (letting $\{\gamma_n\}$ stand for the eigenvalues of B)

$$\begin{aligned}
\operatorname{Tr}(AB) &\leqslant \sqrt{\operatorname{Tr}(A^2)} \cdot \sqrt{\operatorname{Tr}(B^2)} \\
&= \sqrt{\Sigma \alpha_n^2} \cdot \sqrt{\Sigma \gamma_n^2} \\
&< \Sigma \alpha_n \cdot \Sigma \gamma_n \\
&= \operatorname{Tr}(A) \cdot \operatorname{Tr}(B) \\
&\leqslant \left(\frac{\operatorname{Tr}(A) + \operatorname{Tr}(B)}{2} \right)^2.
\end{aligned}$$

(The last inequality is simply the arithmetic mean–geometric mean inequality for two positive real numbers.) This proves (2).

COROLLARY (The Theorem of Yang [3]). *If A, B are two positive definite $n \times n$ matrices then*

(1) $\operatorname{Tr}(AB) > 0$ *and*

(2) $\sqrt{\operatorname{Tr}(AB)} < (\operatorname{Tr}(A) + \operatorname{Tr}(B))/2$.

Proof. Since A, B are finite-dimensional operators they are trivially in the class C_1.

REFERENCES

1. R. BELLMANN, Some inequalities for positive definite matrices in general inequalities, *in* "Proceedings, 2nd Int. Conf. on General Inequalities" (E. F. Beckenbach, Ed.), pp. 89–90, Birkhäuser, Basel, 1980.
2. J. R. RINGROSE, "Compact Non-Self-Adjoint Operators," Van Nostrand–Reinhold, New York, 1971.
3. YISONG YANG, A matrix trace inequality, *J. Math. Anal. Appl.* **133** (1988), 573–574.